RESONANT EXCITATION OF WAVES IN A HEAVY LIQUID BENEATH A VISCOELASTIC PLATE

A. V. Marchenko

UDC 532.59

Nonlinear interactions of waves in a heavy liquid of finite depth at whose surface floats a thin viscoelastic plate modeling ice cover [1] are studied in this work. The source of disturbances in the liquid is a variable external pressure field moving over the ice cover surface. It is well known (see e.g. [2, 3]) that depending in the movement velocity the source of wave excitation with a specific wave number k_0 has a resonant character. In view of this it is of interest to study the resonant excitation of waves in a liquid by an external pressure field which has the form of weakly-modulating wave packets whose spectrum differs from zero in the vicinity of wave numbers $k = mk_0$ $(m \in Z)$ and to study the effect of nonlinearity of the problem on limitation of wave amplitude with resonance.

Resonant interaction of the harmonics of capillary waves at the surface of an ideal liquid was considered in [4-9]. In [10, 11] a study was made of the process of resonant excitation of waves in a heavy liquid with a free surface by a periodic external pressure field.

1. Potential movements of a heavy incompressible liquid with velocity potential φ in dimensionless form are described by the Laplace equation $\Delta\varphi + \partial_{zz}\varphi = 0$ $(\Delta = \partial_{xx} + \partial_{yy})$ with the boundary condition of no flow at the bottom $\partial_z\varphi = 0$, $z = -H$ and dynamic and kinematic conditions at the unknown surface $z = \varepsilon\eta$ [12]:

$$\begin{gathered}\partial_t\eta + \varepsilon\nabla\varphi^s\nabla\eta - \varphi_z^s(1 + \varepsilon^2(\nabla\eta)^2) = 0,\\ \partial_t\varphi^s + (1/2)\,\varepsilon\left[(\nabla\varphi^s)^2 - (\varphi_z^s)^2(1 + \varepsilon^2(\nabla\eta)^2)\right] + p/\rho = 0,\\ \nabla = (\partial_x, \partial_y),\ \varepsilon = a/\lambda \ll 1.\end{gathered} \tag{1.1}$$

Here φ^s, φ_z^s are values of φ and $\partial_z\varphi$ at surface $z = \varepsilon\eta$; p, ρ are liquid pressure and density; a, $2\pi\lambda$ are characteristic amplitude and wavelength. As characteristic values of potential φ, time t, horizontal coordinates x, y, and vertical coordinate z we take, respectively $a\sqrt{g\lambda}$, $\sqrt{\lambda/g}$, λ.

On the liquid surface floats a thin viscoelastic plate modeling ice cover [1]. Pressure in the liquid at the surface under the ice p is connected with external pressure p_0 by the relationship [1]

$$\begin{gathered}(p - p_0)/\rho = \Big(\sum_{\alpha,\beta}(\sigma_{\alpha\beta} - \mu_{\alpha\beta}\partial_t)\,\partial_{\alpha\beta} + D\Delta^2\Big)\,\eta \equiv \widehat{L}\eta,\\ (\alpha, \beta) = (x, y),\quad D = Eh^3/(12(1 - \nu^2)\rho g\lambda^4),\\ \sigma_{\alpha\beta} = h\sigma'_{\alpha\beta}/(\rho g\lambda^2),\quad \mu_{\alpha\beta} = h\mu'_{\alpha\beta}/(\rho\lambda^2\sqrt{g\lambda}),\end{gathered} \tag{1.2}$$

where E, ν are Young's modulus and Poisson's ratio of the plate; $\mu_{\alpha\beta}'$ are plate viscosity coefficients; $\sigma_{\alpha\beta}'$ are tensor components for stresses created in the ice cover by external loads (e.g., by the action of wind); h is plate thickness.

In order to study processes of resonant wave excitation by an external pressure field which has the form of weakly-modulating wave packets we present p_0 in the form

$$p_0/\rho = \sum_{m=-\infty}^{\infty} p_m(X, Y, t)\exp i\theta_m,\quad \theta_m = mk_0x,\quad p_m = p^*_{-m},\ (X, Y) = (\varepsilon x, \varepsilon y)$$

(an asterisk means complex conjugation).

Vladivostok. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 101-109, May-June, 1991. Original article submitted August 8, 1989; revision submitted January 25, 1990.

By expressing the solution of the Laplace equation which satisfies the boundary condition at the bottom in terms of a Fourier integral and assuming that in it z = 0, we find that

$$(\varphi^0, \varphi_z^0) = \int_{-\infty}^{\infty} A(\mathbf{k}, t)(1, s) \exp i\theta d\mathbf{k}. \tag{1.3}$$

Here $s = k \operatorname{th} kH$; $\mathbf{k} = (k_x, k_y)$; $k = |\mathbf{k}|$; $\theta = (\mathbf{kx})$; φ^0, φ_z^0 are values of φ and $\delta_z \varphi$ with z = 0.

We shall find the solution of (1.1) in the form of weakly-modulating wave packets whose Fourier-forms differ from zero in the ε-vicinity of points $\mathbf{k}_m = (mk_0, 0)$. Whence it follows that φ^0, φ_z^0 may be presented as

$$(\eta, \varphi^0, \varphi_z^0) = \sum_{m=-\infty}^{\infty} (\eta_m, \varphi_m^0, \varphi_{z,m}^0) \exp i\theta_m,$$
$$(\eta_m, \varphi_m^0, \varphi_{z,m}^0) = (\eta_{-m}^*, \varphi_{-m}^{0*}, \varphi_{z,-m}^{0*}).$$

Functions η_m, φ_m^0, $\varphi_{z,m}^0$ depend on X, Y, t. From (1.3) we have

$$\varphi_{z,m}^0 = \widehat{F}_m \varphi_m^0, \quad \widehat{F}_m = s_m - i\varepsilon s_{m,1} - (1/2)\varepsilon^2 (s_{m,11}\partial_{xx} + s_{m,22}\partial_{yy}) + O(\varepsilon^3), \tag{1.4}$$

s_m, $s_{m,1}$, $s_{m,11}$, $s_{m,22}$ are values of s, $\partial s/\partial k_x$, $\partial^2 s/\partial k_x^2$, $\partial^2 s/\partial k_y^2$ with $\mathbf{k} = \mathbf{k}_m$.

In Eq. (1.1) there are functions φ^s and φ_z^s which with use of expansion of φ and $\partial_z \varphi$ into a series for powers of ε in the vicinity of z = 0 may be expressed in terms of φ^0, φ_z^0, after which from (1.3) and (1.4) we obtain

$$(\varphi^s, \varphi_z^s) = \sum_{m=-\infty}^{\infty} (\varphi_m, \varphi_{z,m}) \exp(i\theta_m),$$
$$\varphi_{z,m} = \widehat{\Phi}_m(\varphi), \ \widehat{\Phi}_m(\varphi) = \widehat{F}_m \varphi_m + \varepsilon \sum_{l=-\infty}^{\infty} (r_{ml}\eta_l \varphi_{m-l} + \tag{1.5}$$
$$+ i\varepsilon (d_{ml}\eta_l \partial_X \varphi_{m-l} + g_{ml}\varphi_{m-l}\partial_X \eta_l)) + \varepsilon^2 \sum_{l,n=-\infty}^{\infty} f_{mln}\eta_l \eta_n \varphi_{m-l-n} + O(\varepsilon^3).$$

Coefficients $r_{m\ell}$, $d_{m\ell}$, $g_{m\ell}$, $f_{m\ell n}$ depend on the values of k_0, s_m, $s_{m,1}$.

By substituting (1.5) in (1.1) and equating the coefficients with identical harmonics we have with an accuracy to $O(\varepsilon^2)$ an infinite set of equations

$$\partial_t \eta_m - \widehat{F}_m \varphi_m - \varepsilon \sum_{l=-\infty}^{\infty} (r_{ml}^1 \eta_l \varphi_{m-l} + i\varepsilon (d_{ml}^1 \eta_l \partial_X \varphi_{m-l} + g_{ml}^1 \varphi_{m-l} \partial_X \eta_l)) - \varepsilon^2 \sum_{l,n=-\infty}^{\infty} f_{mln}^1 \eta_l \eta_n \varphi_{m-l-n} = 0,$$
$$\partial_t \varphi_m + (\widehat{L}_m^0 + \varepsilon \widehat{L}_m^1 + \varepsilon^2 \widehat{L}_m^2)\eta_m - \varepsilon \sum_{l=-\infty}^{\infty} (r_{ml}^2 \varphi_l \varphi_{m-l} + i\varepsilon d_{ml}^2 \varphi_l \partial_X \varphi_{m-l}) - \varepsilon^2 \sum_{l,n=-\infty}^{\infty} f_{mln}^2 \eta_n \varphi_l \varphi_{m-l-n} + p_m = 0. \tag{1.6}$$

Operators $\widehat{L}_m^n$ are determined from the relationships

$$(\widehat{L} + 1)\eta = \sum_{m,n=-\infty}^{\infty} \varepsilon^n (\widehat{L}_m^n \eta_m) \exp i\theta_m.$$

For example,

$$\widehat{L}_m^0 = \widehat{L}(\mathbf{k}), \ \widehat{L}(\mathbf{k}) = 1 - \sum_{\alpha,\beta} (\sigma_{\alpha\beta} - \mu_{\alpha\beta}\partial_t) k_\alpha k_\beta + D(k_x^2 + k_y^2)^2.$$

Coefficients $f_{m\ell n}^1$, $r_{m\ell}^1$, $d_{m\ell}^1$, $g_{m\ell}^1$ are also expressed in terms of the set k_0, s_m, $s_{m,1}$.

If in the initial instant of time the liquid was at rest, then solution of (1.6) should satisfy the initial conditions

$$\varphi_m = \eta_m = 0, \ t = 0. \tag{1.7}$$

2. We consider the problem of wave generation by a periodic external pressure field moving along the x axis velocity $c = \omega/k_0$. In this case $p_m = p_m' \exp(im\omega t)$, $p'_m = \text{const}$. We make a change in (1.6)

$$\eta_m = \eta_m'(t)\exp(im\omega t), \quad \varphi_m = \varphi_m'(t)\exp(im\omega t)$$

and we introduce new variables

$$u_m = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{L_m}{\omega_m}}\eta_m' + i\sqrt{\frac{\omega_m}{L_m}}\varphi_m'\right), \quad L_m = \widehat{L}(\mathbf{k})\,e^{im\omega t}, \quad \omega_m^2 = L_m s_m.$$

With $\mu_{\alpha\beta} = 0$ system (1.6) with an accuracy up to $O(\varepsilon)$ is written in Hamiltonian form

$$\begin{gathered}\partial_t u_m = -i\partial H/\partial u_m^*,\\ H = \sum_{m=-\infty}^{\infty}\left(\Delta_m u_m u_m^* + g_m(u_{-m} + u_m^*)\right) + \varepsilon\sum_{m,l=-\infty}^{\infty}\left[\alpha_{ml}\left(u_{-m}u_l u_{m-l} + \right.\right.\\ \left.\left. + u_{-m}^* u_l^* u_{m-l}^*\right) + \beta_{ml}\left(u_m u_l^* u_{m-l}^* + u_m^* u_l u_{m-l}\right)\right],\\ \Delta_m = \omega_m - m\omega, \quad g_m = \sqrt{\omega_m/(2L_m)}\,p_m'.\end{gathered} \tag{2.1}$$

Coefficients $\alpha_{m\ell}$, $\beta_{m\ell}$ are determined in terms of the set k_0, s_m, $s_{m,1}$, and ω_m.

It is noted that cubic nonlinear terms in Eqs. (1.6) which have an order of $O(\varepsilon^2)$ cannot be presented in the form of partial derivatives with respect u_m^* and u_m of Hamiltonian H. Therefore, writing of Eqs. (1.6) in canonical form (2.1) is correct with an accuracy up to terms of the order of $O(\varepsilon)$. Canonical variables u_m with $h = 0$ agree with canonical variables of the movement equations for liquid with a free surface found in [13].

Internal resonances are possible in this problem and conditions for the onset of them are determined by the relationships

$$\Delta_m = \Delta_n = 0, \quad m \neq n. \tag{2.2}$$

With internal resonance excitation of the m-th harmonic occurs as a result of nonlinear interaction of the n-th and (m − n)-th harmonics. In liquid with a free boundary condition (2.2) is not fulfilled. Assuming that $n = 1$ and resolving (2.2) with respect to k, we find that

$$k_m = \frac{m\sigma_{xx} + \sqrt{m^2\sigma_{xx}^2 + 4mD(m^2 + m + 1)}}{2mD(m^2 + m + 1)}.$$

In a zero order with respect to ε solution (2.1) satisfying (1.7) is written in the form

$$u_m = ig_m\left(e^{-i\Delta_m t} - 1\right)/\Delta_m. \tag{2.3}$$

In resonance case $\Delta_m = O(\varepsilon)$ solution (2.3) makes sense with $t \lesssim 1$. With long evolution times nonlinear effects limiting the increase in the amplitude of the m-th harmonic due to its self-influence and transfer of energy to other harmonics become marked.

With $\Delta_1 = O(\varepsilon)$, $\Delta_m = O(\varepsilon)$ solution (2.1) is found in the form

$$\begin{gathered}u_j = u_j'(T) + \varepsilon u_j''(t) \quad (j = 1, m), \quad u_k = u_k''(t) + \varepsilon u_k'(T) \quad (k > 0),\\ u_k = u_k(t) \quad (k < 0), \quad T = \varepsilon t,\end{gathered}$$

where $u_k''(t)$ in a zero order with respect to ε is determined by Eqs. (2.3) and there are rapidly oscillating parts of the excitation, but $u_k'(T)$ is found from the equations (in future the primes are omitted)

$$\partial_t u_k = -i\partial\widetilde{H}/\partial u_k^*, \quad \widetilde{H} = H_{m,1} + H_{m,2}, \tag{2.4}$$

$$H_{m,1} = \sum_{k=1}^{\infty}\left(\Delta_k u_k u_k^* + \varepsilon \sum_{l=1}^{k-1} \beta_{kl} u_k^* u_l u_{k-l}\right) + \varepsilon \sum_{k=1}^{m+1}\sum_{l=1}^{k-1} \beta_{kl} u_k u_l^* u_{k-l}^* +$$

$$+ \varepsilon\left(\gamma_1 u_1^* + \gamma_m u_m^* + \gamma_1^* u_1 + \gamma_m^* u_m\right),\ H_{m,2} = \varepsilon \sum_{k=m+2}^{\infty}\sum_{l=1}^{k-1} \beta_{kl} u_k u_l^* u_{k-l}^*,$$

$$\gamma_k = g_k/\varepsilon.$$

System (2.4) has an integral $\tilde{H}$ = const. Phase curves lying in different surfaces of level $\tilde{H}$ cannot intersect and pass from one surface to another. Stationary solutions (2.4) correspond to extreme points of the Hamiltonian

$$\partial\widetilde{H}/\partial u_k^* = 0, \quad k \in Z. \tag{2.5}$$

A state of rest with $\gamma_{1,m} \neq 0$ is not a position of equilibrium system (2.4) and it relates to hypersurface $\tilde{H} = 0$. Emergence into a stationary system from the initial state of rest is only possible in the case when the Hamiltonian has extremes at hypersurface $\tilde{H} = 0$.

Stationary solutions of (2.4) are found in the form of a series

$$u_k = \lim_{l\to\infty} S_{l,k}, \quad S_{l,k} = u_k^0 + \delta_1 u_k + \ldots + \delta_l u_k. \tag{2.6}$$

Here u_k^0 is the accurate solution of the equations $\partial H_{m,1}/\partial u_k^* = 0$; $\delta_\ell u_k$ is solution of system (2.5) linearized with respect to $\delta_\ell u_k$ where the value of $H_{m,1}$ is taken at point $u_k = S_{\ell-1,k} + \delta_\ell u_k$, and $H_{m,2}$ is taken at the point $u_k = S_{\ell-1,k}$. A criterion for applicability of this method is convergence of series (2.6).

For an example we consider the case $\Delta_1 = \varepsilon\Delta$, $\Delta = O(1)$, $\Delta_k = O(1)$. It follows from (2.5) that u_1^0 is a root of the equation $\Delta u_1 + \beta_{21}^2|u_1|^2 u_1/\Delta_2 + \gamma_1 = 0$. With $\Delta = 0$ we find that $|u_1| = \gamma^{1/3}$, $\gamma = |\Delta_2\gamma_1/\beta_{21}^2|$. With $\gamma \ll 1$ stationary solution (2.4) is obtained in the form of a series with respect to powers of $\gamma^{1/3}$. Calculations show that $\tilde{H} \neq 0$ for this solution. Therefore, emergence into the stationary regime constructed from the original state of rest is impossible.

With $\Delta_{1,m} = O(\varepsilon)$ the solution of (2.4) has the following structure:

$$|u_k| = O\left(\varepsilon^{\nu_k}\right), \quad \nu_k = \min\,(k - m\,[k/m],$$
$$m\,([k/m] + 1) - k)\ (k > m),\ \nu_k = \min\,(k - 1,\ m - k)\ (1 < k < m,\ m > 0). \tag{2.7}$$

In particular, in the absence of internal resonances $|u_k| = O(\varepsilon^{k-1})$ is fulfilled. With resonance in the 2nd harmonic (m = 2) we have $|u_k| = O(\varepsilon^{k-2})$, $k \geq 2$.

It follows from (2.7) that from (2.4) it is possible with an accuracy up to $O(\varepsilon^2)$ to exclude all terms apart from u_1 and u_m interacting resonantly. With $m > 2$ in equations for interaction of u_1 and u_m there will be cubic nonlinear terms which have the order of $O(\varepsilon^2)$. It is noted that in deriving Eq. (2.1) from (2.4) cubic terms were not considered.

3. Equations for interaction of the 1st and m-th harmonics are derived from (1.6) using property (2.7). With $m > 2$ with an accuracy up to $O(\varepsilon^2)$ we obtain

$$\left(i\widehat{D}_{1,j} + \varepsilon\widehat{D}_{2,j} + \widetilde{\Delta}_j\right)\varphi_j = \varepsilon\left(\left(\varkappa_{j1}|\varphi_1|^2 + \varkappa_{jm}|\varphi_m|^2 + \alpha_j\partial_T\varphi_0 + \beta_j\partial_X\varphi_0\right)\varphi_j +\right.$$
$$\left. + \delta_{3m}\nu_j F_j\right) + \widetilde{p}_j \quad (j = 1, m),\ F_1 = \varphi_1^{*2}\varphi_3,\ F_3 = \varphi_1^3,$$
$$(\partial_{TT} - H\Delta)\varphi_0 = \widehat{D}_{0,1}|\varphi_1|^2 + \widehat{D}_{0,m}|\varphi_m|^2,$$
$$\widehat{D}_{1,j} = (1 + i\mu_j)\,\partial_T + \mathbf{V}_j\cdot\nabla,$$
$$\widehat{D}_{2,j} = W_{j,XX}\partial_{XX} + 2W_{j,XY}\partial_{XY} + W_{j,YY}\partial_{YY} - \frac{1}{2j\omega}\partial_{TT} - i\,U_j\partial_{XT}, \tag{3.1}$$
$$\widehat{D}_{0,j} = \left(j^2k_0^2 - s_j^2\right)\partial_T - 2k_0\omega^{-1}s_j\partial_X, \quad \mathbf{V}_j = \frac{1}{2\omega}\nabla_k(Ls)|_{\mathbf{k}=\mathbf{k}_j},$$
$$W_{j,\alpha\beta} = \frac{1}{4j\omega}\left.\frac{\partial^2(Ls)}{\partial k_\alpha\,\partial k_\beta}\right|_{\mathbf{k}=\mathbf{k}_j}, \quad U_j = -\mu_{xx}k_0(s_m + mk_0 s_{m,1})/\omega,$$
$$\widetilde{\Delta}_j = \left(j^2\omega^2 - \omega_j^2\right)/(2j\varepsilon\omega),$$
$$\widetilde{p}_j = (\varepsilon\partial_T p_j - ij\omega p_j)/(2j\varepsilon\omega).$$

Equations for interaction of the 1st and 2nd harmonics with an accuracy up to $O(\varepsilon)$ have the form

$$(\widehat{D}_{1,j} - i\widetilde{\Delta}_j)\varphi_j = T_j N_j - i\widetilde{p}_j \quad (j = 1, 2),$$
$$(\partial_{TT} - H\Delta)\varphi_0 = \widehat{D}_{0,1}|\varphi_1|^2 + \widehat{D}_{0,2}|\varphi_2|^2,$$
$$N_1 = \varphi_1^*\varphi_2, \quad N_2 = \varphi_1^2.$$

In view of their cumbersome nature expressions for the coefficients κ_{ij}, α_j, β_j, ν_j, T_j are not provided.

If plate thickness h tends towards zero, then fulfillment of the condition for generation of high harmonics is impossible. Equations (3.1) are converted into a set of Davey–Stewartson equations [8, 9]. With $\mu_{\alpha\beta} = \sigma_{\alpha\beta} = 0$, $H \to \infty$, Eqs. (3.1) degenerate into a nonlinear Schroedinger equation

$$i(\partial_T + \omega'\partial_X)\varphi_1 + (\varepsilon/2)(\omega''\partial_{XX} + \omega' k_0^{-1}\partial_{YY})\varphi_1 + \widetilde{\Delta}_1\varphi_1 = \varepsilon\varkappa|\varphi_1|^2\varphi_1 + \widetilde{p}_1,$$
$$\varkappa = k_0^4(2 - 13Dk_0^4)/(\omega(1 - 14Dk_0^4)). \quad (3.2)$$

With $\tilde{p}_1 = \tilde{\Delta}_1 = 0$, $\kappa\omega'' < 0$ periodic solution (3.2) is unstable, which leads to decay of the wave envelope into individual solitons. Soliton solutions (3.2) in the unidimensional case are written as

$$\varphi_1 = \psi(\zeta)\exp[i(rX + sT)], \ \zeta = X - vT,$$
$$\psi = A\,\mathrm{ch}^{-1}(B\zeta), \ v = \omega' + \varepsilon r\omega'',$$
$$A = \sqrt{-2R/(\varepsilon\varkappa)}, \ B = \sqrt{2R/(\varepsilon\omega'')}, \ R = s + r\omega' + \varepsilon r^2\omega''/2.$$

In the case of conformity of the soliton envelope velocity with phase velocity ω/k_0 for the wave running beneath it, wave packet $\varphi = \varphi_1\exp[i(\theta_1 - \omega t)]$ is a soliton with an oscillating structure. In a zero order with respect to ε this condition is fulfilled with $\omega' = \omega/k_0$. By solving this relationship with respect to k_0 we find that $k_0 = (3D)^{-1/4}$.

4. We study the effect of low plate viscosity on development of vibrations in an infinitely deep liquid excited by periodic external pressure at its surface. We consider the case of absence of internal resonances, i.e.,

$$\Delta_1 = \varepsilon(\Delta + i\mu) = O(\varepsilon), \ \Delta_k = O(1), \ k \neq 1 \quad (\mu = k_0^2\mu_x/(2\varepsilon),$$

Δ is frequency detuning). From (3.2) we find that this process is described by the equation

$$i\partial_T\varphi + (\Delta + i\mu)\varphi = \varepsilon\varkappa|\varphi|^2\varphi + p, \ p \equiv \widetilde{p}_1 = \mathrm{const}, \ \varphi \equiv \varphi_1, \quad (4.1)$$

which with $\mu = 0$ has the integral

$$H = -\Delta(\Phi^2 + \psi^2) + \varepsilon\varkappa(\Phi^2 + \psi^2)^2/2 + 2p\Phi, \ \varphi = \Phi + i\psi.$$

We introduce the notation $E = \Phi^2 + \psi^2$. From (4.1) with $\mu = 0$ we find that

$$\partial_T E = \sqrt{P_4(E)}, \ P_4(E) = -H^2 + 2E(2p^2 - \Delta H) +$$
$$+ E^2(\varepsilon\varkappa H - \Delta^2) + \varepsilon\Delta\varkappa E^3 - \varepsilon^2\varkappa^2E^4/4.$$

The solution of the equation $E = E(T)$ obtained is expressed in terms of elliptical functions. Constant H is selected from the initial conditions with $t = 0$. We note that $P_4(E) \geq 0$ with $t = 0$. With large E there is fulfillment of $P_4(E) < 0$. Therefore for the equation $P_4(E) = 0$ there is always a positive root. Movement with any initial conditions occurs in a limited region in phase plane (Φ, ψ). For Eq. (4.1) with $\mu \ll 1$ there are singular points in plane (Φ, ψ) which are found from

$$\varepsilon\varkappa\Phi^3 - \Delta\Phi + p = O(\mu^2), \ \psi = \mu\Phi/(\varepsilon\varkappa\Phi^2 - \Delta). \quad (4.2)$$

With $D = 4\Delta^3/(27\varepsilon\kappa) - p^2 > 0$ the first equation of (4.2) has three real roots which are written in the form

$$\Phi_i = \frac{2}{\sqrt{3}}\sqrt{\frac{\Delta}{\varepsilon\varkappa}}\sin(\chi + 2(i-1)\pi/3), \quad \chi = \arcsin\left(\frac{3\sqrt{3}\,p\sqrt{\varepsilon\varkappa}}{\Delta\sqrt{\Delta}}\right), \; i = 1-3.$$

With $D = 0$ there are two real roots one of which is twofold $\Phi_2 = \Phi_3 = \sqrt{\frac{\Delta}{3\varepsilon\varkappa}}$, $\Phi_1 = -2\sqrt{\frac{\Delta}{3\varepsilon\varkappa}}$, and with $D < 0$ there is only one real root. For example, with exact resonance $\Phi_1{}^0 = -(p/(\varepsilon\kappa))^{1/3}$, $\Delta = 0$. Each singular point relates to stationary solution (4.1). Point $\Phi_1{}^0$ with $\Delta = 0$ corresponds to the stationary resonant solution obtained in Sec. 2 in the form of a series with powers of $\gamma^{1/3}$.

We determine the stability of the stationary solutions. By linearizing Eq. (4.1) in the vicinity of singular points, with an accuracy up to $O(\mu)$ we find that

$$x_t = X^i x, \; x = (x_1, x_2), \; \Phi = \Phi_i + x_1, \; \psi = \psi_i + x_2,$$
$$X^i_{jj} = \mu\left(\frac{2\varepsilon\varkappa\Phi_i^2(-1)^{i-1}}{2\varepsilon\varkappa\Phi_i^2 - \Delta} - 1\right), \quad X^i_{jk} = (-1)^k(\varepsilon\varkappa(2 + (-1)^j)\Phi_i^2 - \Delta)$$
$$(j, k = 1, 2).$$

The eigenvalues of matrices X^i are

$$\lambda_i^{\pm} = -\mu \pm i\sqrt{(\varepsilon\varkappa\Phi_i^2 - \Delta)(3\varepsilon\varkappa\Phi_i^2 - \Delta)} + O(\mu^2). \qquad (4.3)$$

It follows from (4.3) that with $D < 0$ a singular point is a stable focus and the solution relating to it is asymptotically stable. With $D = 0$ this singular point is the stable focus, and another is the stable node. Both stationary solutions in this case are asymptotically stable. With $0 < D < D_1$, where D_1 is determined from the condition $\lambda_2{}^+ = 0$, yet another singular point appears, i.e., a stable node. The solution corresponding to it is asymptotically stable. With $D > D_1$ there are three singular points: a stable focus, a stable node, and a saddle point. Solutions relating to the focus and the node are asymptotically stable, but the solution corresponding to the saddle point is unstable. It is noted that with $\mu = 0$ singular points $\Phi_{1,3}$ are centers and according to Lyapunov the solutions relating to them are stable.

A wave number $k = (2/(13D))^{1/4}$ exists with which $\kappa = 0$ and solution (4.1), which satisfies the zero initial conditions, takes the form $\varphi = \frac{p}{\Delta + i\mu}(e^{(i\Delta-\mu)t} - 1)$. With $t \to \infty$ this solution emerges into a stationary regime $\varphi_s = -p/(\Delta + i\mu)$ and with small Δ, μ it has the maximum amplitude.

5. In an infinitely deep liquid beneath an elastic plate interactions of the 1st and 2nd harmonics in the unidimensional case with $\mu_{\alpha\beta} = \sigma_{\alpha\beta} = p_{1,2} = \Delta_{1,2} = 0$ are described by the equations

$$\begin{aligned} &\partial_T\varphi_1 + \omega_1'\partial_X\varphi_1 = k_2^2\varphi_1^*\varphi_2, \\ &\partial_T\varphi_2 + \omega_2'\partial_X\varphi_2 = (1/2)k_2^2\varphi_1^2, \quad k_2 = (14D)^{-1/4}. \end{aligned} \qquad (5.1)$$

Equations (5.1) have an integral

$$\int_{-\infty}^{\infty}(|\varphi_1|^2 + 2|\varphi_2|^2)\,dX = \text{const}, \quad \varphi_{1,2} \to 0, \; |X| \to \infty,$$

from which it follows that if in the initial instant of time all of the wave energy is concentrated in the 1st harmonic then with the passage of time the energy of harmonics becomes comparable in value. The process of energy transfer between harmonics of a periodic nature is described by elliptical functions [14].

Equations for interaction of the 1st and 2nd harmonics in the unidimensional case with $\mu_{\alpha\beta} = \sigma_{\alpha\beta} = p_{1,m} = \Delta_{1,m} = 0$ look as follows [15]:

$$i\left(\partial_T\varphi_j + \omega_j'\partial_X\varphi_j\right) + (1/2)\,\varepsilon\omega_j''\partial_{XX}\varphi_j = \varepsilon\widetilde{N}_j,$$
$$\widetilde{N}_j = (\varkappa_{j1}|\varphi_1|^2 + \varkappa_{jm}|\varphi_m|^2)\,\varphi_j + \delta_{m3}\nu_j F_j, \tag{5.2}$$
$$j = 1, m, \quad \nu_1 = -\frac{177}{390D}, \quad \nu_3 = -\frac{27}{65D}.$$

Equations (5.2) with m = 3 have an integral

$$\int_{-\infty}^{\infty} (|\varphi_1|^2 + (\nu_1/\nu_3)|\varphi_3|^2)\,dX = \text{const}, \quad \varphi_{1,3}\to 0, \quad |X|\to\infty. \tag{5.3}$$

With m > 3 from (5.2) it follows that

$$\int_{-\infty}^{\infty} |\varphi_j|^2\,dX = \text{const}, \quad \varphi_j \to 0, \quad |X|\to\infty.$$

It follows from (5.3) that interaction of the 1st and 3rd harmonics exhibits a property similar to interaction of the 1st and 2nd harmonics. If in the initial instant of time all of the wave energy is concentrated in the 1st harmonic, then with the passage of time the energies of the 1st and 3rd harmonics become comparable in value. The difference is in the characteristic times of the processes. With interaction of the 1st and 2nd harmonics the characteristic time of transfer of energy is of the order of $O(\varepsilon^{-1})$, and with interaction of the 1st and 3rd harmonics it is of the order of $O(\varepsilon^{-2})$. In the case of interaction of periodic waves [the dependence of X in (5.2) is absent] solution (5.2) is expressed in terms of elliptical functions by the equations

$$\varphi_j = \sqrt{-\nu_j}z_j\exp(if_j) \quad (j = 1, 3), \quad z_1^2 + z_3^2 = C^2 = \text{const},$$
$$P_3(z_1) + \nu_1\sqrt{\nu_1\nu_3}z_1^2z_3\cos\gamma = 0, \quad \gamma = f_3 - 3f_1,$$
$$P_3(z_1) = (2\varkappa_{13}\nu_3 - 2\varkappa_{11}\nu_1 + \varkappa_{31}\nu_1 - \varkappa_{33}\nu_3)\,z_1^3/3 + (\varkappa_{33}\nu_3 - 2\varkappa_{13}\nu_3)\,C^2z_1 + \widetilde{C},$$
$$\widetilde{C} = \text{const},$$
$$z_1' = \sqrt{P_6(z_1)}, \quad f_j' = (\varkappa_{j1}\nu_j z_1^2 + \varkappa_{j3}\nu_3 z_3^2) +$$
$$+ \nu_1\sqrt{\nu_1\nu_3}\cos\gamma\, z_1^j z_3^{2-j}, \quad P_6(z_1) = \nu_1^3\nu_3 z_1^4 z_3^2 + P_3^2(z_1).$$

If $P_6(z_1)$ has a twofold root $z_1 = 0$ and $\partial^2P_6/\partial z_1^2 > 0$ with $z_1 = 0$, while $P_6(z) > 0$ with $0 < z < z_1(0)$, then $z_1 \to 0$ with $t \to \infty$. In other words, with the passage of time there is total transfer of energy to the 3rd harmonic.

With resonance in the m-th (m > 3) harmonic the transfer of energy does not occur in this approximation. If in the initial instant of time the m-th harmonic is small compared with the first, then it ceases with evolution times of $t = O(\varepsilon^{-2})$.

With resonance in the m-th ($m \geq 2$) harmonic nonlinear terms of Eq. (1.1) with substitutions (1.5) responsible for exchange of energies between harmonics (amplitude interaction) have the order $O(\varepsilon^m)$. Terms responsible for nonlinear phase interaction and self-influence of harmonics have the order $O(\varepsilon^3)$. Phase interaction and self-influence leads to a nonlinear shift in the frequency of interacting waves $\Delta\omega_j = \varepsilon^2(\varkappa_{j1}|\varphi_1|^2 + \varkappa_{jm}|\varphi_m|^2)$ and retention of their energy. With m = 3 terms of the same order of smallness relate to processes of amplitude and phase interaction in Eqs. (5.2). With m > 3 phase interaction and self-influence predominate.

LITERATURE CITED

1. Yu. P. Doronin and D. E. Kheisin, Marine Ice [in Russian], Gidrometeoizdat, Leningrad (1975).
2. V. V. Bogorodskii, V. P. Gavrilo, and O. A. Nedoshivin, Breaking of Ice: Methods, Technical Means [in Russian], Gidrometeoizdat, Leningrad (1983).

3. R. M. S. M. Schulkes and A. D. Sneyd, "Time-dependent response of floating ice to a steadily moving load," J. Fluid Mech., 186, 25 (1988).
4. L. F. Mcgoldrick, "On Wilton's ripples: a special case of resonant interactions," J. Fluid Mech., 42, No. 4 (1970).
5. L. F. Mcgoldrick, "An experiment on second-order capillary gravity resonant wave interactions," J. Fluid Mech., 40, No. 2 (1970).
6. L. F. Mcgoldrick, "On the rippling of small waves: a harmonic nonlinear nearly resonant interaction," J. Fluid Mech., 52, No. 4 (1972).
7. V. D. Djordjevic and L. G. Redekopp, "On two-dimensional packets of capillary-gravity waves," J. Fluid Mech., 79, No. 4 (1977).
8. A. Davey and K. Stewartson, "On three-dimensional packets of surface waves," Proc. R. Soc. London A, 388, 101 (1974).
9. M. J. Ablowits and H. Sigur, Solitons and Methods of the Inverse Scattering Transform, Society for Industrial and Applied Mathematics, Philadelphia (1981).
10. S. V. Nesterov, "Excitation of waves of finite amplitude by a running pressure system," Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana, 4, No. 10 (1986).
11. A. V. Marchenko and N. R. Sibgatullin, "Resonant excitation of long waves in a two-layer liquid by a variable pressure at the free surface," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaz., No. 2 (1990).
12. G. Ewen and B. Lake, Nonlinear Dynamics of Gravitational Waves in Deep Water [Russian translation], Mir, Moscow (1987).
13. V. E. Zakharov, "Stability of periodic waves of finite amplitude at the surface of a deep liquid," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1968).
14. A. V. Marchenko and N. R. Sibgatullin, "Resonant interaction of waves in a heavy liquid located beneath an elastic plate," Vestn. Mosk. Gos. Univ., Ser. 1, Mat. Mekh., No. 4 (1986).
15. A. T. Il'ichev and A. V. Marchanko, "Propagation of long nonlinear waves in a heavy liquid beneath an ice cover," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaz., No. 1 (1989).

THERMOPHORETIC MOTION OF AN ENSEMBLE OF MODERATELY COARSE AEROSOL PARTICLES

M. A. Gaidukov, V. A. Kostruba, and A. V. Terzyan

UDC 533.72+541.182

Knowledge of the laws governing the behavior of an ensemble of aerosol particles in a nonisothermal gas makes it possible to increase the efficiency of many industrial operations (production of powders, removal of valuable or hazardous by-products from the atmosphere, etc.). Such knowledge can also be useful in developing both natural and artificial methods of influencing cloud formation and movement. The latter is important, for example, in the use of aerosols in agriculture.

The solution of thermophoresis problems entails calculation of the relative motion of a nonuniformly heated gas and aerosol particles suspended in it. The principal assumption underlying the hydrodynamic method of calculation proposed in [1] is that the particles are distant from one another and can each be regarded as an individual particle located in an infinite gas. Gaidukov and Melekhov [2] and Yalamov et al. [3] used this method to develop an approach which makes it possible to study the thermophoretic motion of an arbitrary collection of solid aerosol particles located close enough to one another to allow their hydrodynamic interaction. By hydrodynamic interaction, it is meant that the interaction is due to the fact that a particle moving in the medium generates a velocity field that affects the motion of other particles. By virtue of the assumptions made in the mathematical formulation of the problem, the results presented in [2, 3] are valid only for an ensemble con-

Erevan. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 109-115, May-June, 1991. Original article submitted November 11, 1987; revision submitted June 23, 1989.

 0021-8944/91/3203-0402$12.50